SET CONCEPTS II

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ABSTRACT. We continue discussing the basic concepts of set theory. In this installment, we discuss functions.

1. FUNCTIONS

Let A and B be sets. A *function* from A to B is a subset $f \subset A \times B$ of the cartesian product of A with B such that for every $a \in A$ there exists a unique $b \in B$ such $(a, b) \in f$. This is the technical definition; think about how it relates to the intuitive approach below.

Intuitively, a *function* from a set A to a set B is an assignment of every element in A to some element in B. Another way of describing this is that we think of a function as a kind of vehicle, something which sends each element of A to an element of B. If we think of elements are the nouns of set theory and sets as the adjectives (an element has a property if it is in the set of things with that property), then we may think of functions as the verbs.

There are many familiar examples of functions from the set of real numbers into itself, for example, sin, cos, log, and so forth. It is essential in mathematics, and extremely useful as a way of thinking in general, to expand our view of functions so that they can send elements from any set to any other set.

Let f be a function from A to B. If $a \in A$, the element of B to which a is assigned by f is denoted $f(a)$; in other words, the place in B to which a is sent by f is denoted $f(a)$. We declare that a function must satisfy the following "defining property":

$$\forall a \in A \exists! b \in B \ni f(a) = b.$$

In words, for every element a in A there exists a *unique* element b in B such that a is sent to b by f.

If f is a function from A to B, this fact is denoted

$$f : A \to B.$$

We say that f *maps* A *into* B , and that f is a function *on* A. For this reason, functions are sometimes called *maps* or *mappings* . If $f(a) = b$, we say that a is *mapped to* b by f. We may indicate this by writing $a \mapsto b$.

Two functions $f : A \to B$ and $g : A \to B$ are considered *equal* if they act the same way on every element of A:

$$f = g \Leftrightarrow (a \in A \Rightarrow f(a) = g(a)).$$

Thus to show that two functions f and g are equal, select an arbitrary element $a \in A$ and show that $f(a) = g(a)$.

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If A is sufficiently small, we may explictly describe the function by listing the elements of A and where they go; for example, if $A = \{1, 2, 3\}$ and $B = \mathbb{R}$, a perfectly good function is described by $\{1 \mapsto 23.432, 2 \mapsto \pi, 3 \mapsto \sqrt{593}\}$.

However, if A is large, the functions which are easiest to understand are those which are specified by some *rule* or *algorithm* . The common functions of single variable calculus are of this nature.

Example 1. Let $\mathbb{R}$ be the set of real numbers. The following are all functions from $\mathbb{R}$ into $\mathbb{R}$:

- $f(x) = 0$;
- $f(x) = x$;
- $f(x) = x^3 + 3x + 17$;
- $f(x) = \sin(x)$;
- $f(x) = \exp(x)$.

The following are functions from the set of positive real numbers into $\mathbb{R}$:

- $f(x) = \sqrt{x}$;
- $f(x) = \log(x)$.

Note that $\tan(x)$ is not a function from $\mathbb{R}$ into $\mathbb{R}$, because it is not defined at (for example) the point $\frac{\pi}{2} \in \mathbb{R}$. □

Some functions are contructed from existing functions by specifying cases.

Example 2. Let $\mathbb{R}$ be the set of real numbers. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x < 0; \\ x^3 & \text{if } x \geq 0. \end{cases}$$

The reader familiar with calculus may ask himself whether or not the first, second, and third derivatives exist and are continuous for this function. □

Example 3. Let X be a set and let $A \subset X$. The *characteristic function* of A in X is a function $\chi_A : X \to \{0, 1\}$ defined by

$$\chi_A(x) = \begin{cases} 0 \text{ if } x \notin A; \\ 1 \text{ if } x \in A. \end{cases}$$

In particular, let $X = [0, 1] \subset \mathbb{R}$ be the closed unit interval and let $A = \mathbb{Q} \cap X$ be the set of rational numbers in this interval. Think about the graph of the function χ_A. □

Example 4. Suppose we designed a computer system that records information on patients in a hospital. Each patient is assigned a number upon admission, which is just the next available number, starting with zero. We create a program which allows the user to type a working diagnosis of 60 characters or less for this patient, and file this information under the patient number. We only allow the user to type capital letters, spaces, commas, and periods in this diagnosis. The file may be viewed as a function

$$\text{DIAG(patient number)} = \text{“patient diagnosis”};$$

here, DIAG : $\mathbb{N} \to B$, where B is the set of all possible strings of allowed characters with length less than or equal to 60 which can be typed on a computer keyboard. The size of B is 29^{60} (why?). □

2. Images and Preimages

If $f : A \to B$, the set A is called the *domain* of the function and the set B is called the *codomain* . We often think of a function as taking the domain A and placing it in the codomain B. However, when it does so, we must realize that more than one element of A can be sent to a given element in B, and that there may be some elements in B to which no elements of A are sent.

If $C \subset A$, we define the *image* of C under f to be the set

$$f[C] = \{b \in B \mid f(a) = b \text{ for some } a \in A\}.$$

The image of the domain is called the *range* of the function.

A function $f : A \to B$ is called *surjective* (or *onto*) if

$$\forall b \in B \exists a \in A \ni f(a) = b.$$

Equivalently, f is surjective if $f[A] = B$.

If $D \subset B$, we define the *preimage* of D under f to be the set

$$f^{-1}[D] = \{a \in A \mid f(a) \in D\}.$$

If D is a singleton set, that is if $D = \{b\}$ for some $b \in B$, we may write $f^{-1}[b]$ instead of $f^{-1}[\{b\}]$.

A function $f : A \to B$ is called *injective* (or *one-to-one*) if

$$\forall a, b \in A, f(a) = f(b) \Rightarrow a = b.$$

Equivalently, f is injective if for all $b \in B$, $f^{-1}[b]$ contains at most one element in A.

A function $f : A \to B$ is called *bijective* if it is both injective and surjective. Such a function sets up a *correspondence* between the elements of A and the elements of B.

Example 5. First we consider "real-valued functions of a real variable". This simply means that the domain and the codomain of the function is $\mathbb{R}$.

- $f(x) = x^3$ is bijective;
- $g(x) = x^2$ is neither injective nor surjective;
- $h(x) = x^3 - 2x^2 - x + 2$ is surjective but not injective;
- $a(x) = \arctan(x)$ is injective but not surjective.

Let $A = \{-1, 1, 2\}$. Some of the images and preimages of A are:

- $f[A] = \{-1, 1, 8\}$;
- $g[A] = \{1, 4\}$;
- $h[A] = \{0\}$;
- $f^{-1}[A] = \{-1, 0, \sqrt[3]{2}\}$;
- $g^{-1}[A] = \{-\sqrt[3]{2}, -1, 1, \sqrt[3]{2}\}$;
- $a^{-1}[A] = \varnothing$.

Example 6. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \sin x$. The set $A = \{\frac{k\pi}{3} \mid k \in \mathbb{Z}\}$ is the set of multiples of $\frac{\pi}{3}$. The image of this set under f is

$$f[A] = \{0, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\}.$$

Let $B = \{0, \frac{1}{2}\}$. The preimage of this set under f is

$$f^{-1}[B] = \{k\pi \pm \frac{\pi}{6} \mid k \in \mathbb{Z}\}.$$

This function is not surjective, because there are points in $\mathbb{R}$ which are not the sine of any angle, and it is not injective, since more than one point is mapped to a given point in the range. $\square$

Example 7. Let $\mathbb{N}$ be the set of natural numbers and let $\mathbb{Z}$ be the set of integers. The function $f : \mathbb{Z} \to \mathbb{Z}$ given by $n \mapsto 2n$ is injective but not surjective.

The function $g : \mathbb{Z} \to \mathbb{N}$ given by $n \mapsto \sqrt{n^2}$ is surjective but not injective. The preimage of $5 \in \mathbb{N}$ under g is $\{-5, 5\}$.

The function $h : \mathbb{Z} \to \mathbb{Z}$ given by $n \mapsto -n$ is bijective. $\square$

Example 8. Let A be the set of all animals in a zoo and let B be the set of all species of animals on earth. Then we obtain a function $f : A \to B$ by defining $f(a) = b$, where the species of animal a is b. This function is surjective only if this is an unbelievably excellent (and large) zoo, for this would mean it has at least one animal of every species on earth. It is injective only if every animal is very lonely, for this would mean that the zoo contains at most one animal of a given species.

However, a function which assigns to every animal on Noah's Ark its species would be surjective but not injective, since he had two of every kind. Such a function is sometimes called "two-to-one". $\square$

Example 9. If DIAG is a function which assigns to a patient his diagnosis, then DIAG is injective unless two patients have the same diagnosis. It is not surjective unless we have admitted at least 29^{60} patients. $\square$

The *graph* of a function $f : A \to B$ is defined to be

$$\{(a, b) \in A \times B \mid b = f(a)\}.$$

Example 10. Let $\mathbb{R}$ denote the set of real numbers. Recall that $\mathbb{R}^n$ is the set of ordered n-tuples of real numbers. This set may be called *n-dimensional space* . Thus $\mathbb{R}^2$ is a plane and $\mathbb{R}^3$ is three-dimensional space. We consider functions defined on multidimensional space. Note that we identify $\mathbb{R}^m \times \mathbb{R}^n$ with $\mathbb{R}^{m+n}$. Thus the graph of a function $f : \mathbb{R}^m \to \mathbb{R}^n$ is

$$\{(x_1, \ldots, x_m, y_1, \ldots, y_n) \mid f(x_1, \ldots, x_m) = (y_1, \ldots, y_n)\}.$$

For example, the graph of a differentiable function $f : \mathbb{R} \to \mathbb{R}$ is a curve in $\mathbb{R}^2$ and the graph of a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ is a surface in $\mathbb{R}^3$. $\square$

3. Composition of Functions

Let A, B, and C be sets and let $f : A \to B$ and $g : B \to C$. The *composition* of f and g is the function

$$g \circ f : A \to C$$

given by

$$g \circ f(a) = g(f(a)).$$

The domain of $g \circ f$ is A and the codomain is C. The range of $g \circ f$ is the image under g of the image under f of the domain of f.

Example 11. Let A be the set of living things on earth, B the set of species, and C be the set of positive real numbers. Let $f : A \to B$ assign each living thing to its species, and let $g : B \to C$ assign each species to its average mass. Then $g \circ f$ guesses the mass of a living thing. □

If f and g are injective, then $g \circ f$ is injective. If f and g are surjective, then $g \circ f$ is surjective. For example, we prove the first of these statements.

Proposition 1. *Let A, B and C be sets and let $f : A \to B$ and $g : B \to C$ be injective functions. Then $g \circ f$ is injective.*

Proof. To show that a function is injective, we select two elements in the domain and assume that they are sent to the same place; it then suffices to show that they were originally the same element.

Let $h = g \circ f$. Let $a_1, a_2 \in A$ and suppose that $h(a_1) = h(a_2) = c$. Let $b_1 = f(a_1)$ and let $b_2 = f(a_2)$. Since $h(a) = g(f(a))$ for each $a \in A$, we have $g(f(a_1)) = g(b_1)$ and $g(f(a_2)) = g(b_2)$. Thus $g(b_1) = g(b_2) = c$. Since g is injective, it follows that $b_1 = b_2$ by the definition of injectivity. Since f is injective, it follows that $a_1 = a_2$, again by definition. □

Example 12. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$ and let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) = \sin x$. Then $g \circ f : \mathbb{R} \to \mathbb{R}$ is given by $g \circ f(x) = \sin x^2$ and $f \circ g : \mathbb{R} \to \mathbb{R}$ is given by $f \circ g(x) = \sin^2 x$. □

This example demonstrates that composition of functions is not a commutative operation. However, the next proposition tells us that composition of functions is associative.

Proposition 2. *Let A, B, C, and D be sets and let $f : A \to B$, $g : B \to C$, and $h : C \to D$ be functions. Then $h \circ (g \circ f) = (h \circ g) \circ f$.*

Proof. To show that two functions are equal, it suffices to show that they act the same way on an arbitrary element of the domain.

Let $a \in A$. Then

$$h \circ (g \circ f)(a) = h(g \circ f(a)) = h(g(f(a)) = h \circ g(f(a) = (h \circ g) \circ f(a).$$

□

4. Restrictions and Bijections

Let $f : X \to Y$ be a function and let $Z = f(X)$ be the range of f. The same function f can be viewed as a function $f : X \to Z$. It is standard in this case to call the function, viewed in this way, by the same name. Note that the function $f : X \to Z$ is surjective. Thus any function is a surjective function onto its range.

Let $f : X \to Y$ be a function and let $A \subset X$ be a subset of the domain of f. The *restriction* of f to A is a function

$$f \restriction_A : A \to Y \text{ given by } f \restriction_A (a) = f(a).$$

Thus given any function and any subset of the domain, there is a function which coincides with the original one, but whose domain is the subset. For example, the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ can certainly be viewed as a function on the integers, sending each integer to its square.

Notice that restriction of a function to a subset of the domain does not necessarily effect the range. For example, if $f : \mathbb{R} \to \mathbb{R}$ is the sine function $f(x) = \sin x$, then the range of $f \restriction_{[0,2\pi]}$ is the same as the range of f on the entire real line.

However, if the original function is injective, then so is any restriction of it.

Let A be any set. The *identity function* on A if the function $\mathrm{id}_A : A \to A$ given by $\mathrm{id}_A(a) = a$ for every $a \in A$. Thus the identity function on A is that function which does nothing to A.

Let $f : A \to B$ be a function. We say that f is *invertible* if there exists a function $g : B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. In this case we call g the *inverse* of f. The inverse of a function f is often denoted f^{-1}.

Proposition 3. *Let $f : A \to B$ be a function. Then f is invertible if and only if f is bijective.*

Proof.

($\Rightarrow$) Suppose that f is invertible and let g be an inverse for f. We wish to show that f is bijective, so we show that f is both injective and surjective.

To show surjectivity, select an arbitrary element $b \in B$ and find an element $a \in A$ such that $f(a) = b$. We let $a = g(b)$. Thus $f(a) = f(g(b)) = \mathrm{id}_B(b) = b$. This shows that f is surjective.

To show injectivity, select to arbitrary elements $a_1, a_2 \in A$ and assume that $f(a_1) = f(a_2)$. Now it suffices to show that $a_1 = a_2$. Since $f(a_1) = f(a_2)$, we have $g(f(a_1)) = g(f(a_2))$. Thus $\mathrm{id}_A(a_1) = \mathrm{id}_A(a_2)$. But this implies that $a_1 = a_2$, so that f is injective.

($\Leftarrow$) Suppose that f is bijective. We wish to show that f is invertible. Let $b \in B$. Since f is surjective, there exists $a \in A$ such that $f(a) = b$. Since f is injective, a is unique with this property. Define $g(b) = a$. Since b was arbitrary, this defines a function $g : B \to A$.

Now $f(g(b)) = f(a) = b$, so $f \circ g = \mathrm{id}_B$. Also $g(f(a)) = g(b) = a$, so $g \circ f = \mathrm{id}_A$. This completes the proof. □

Let X be a set. A *permutation* of X is a bijective function $\phi : X \to X$. The set of of permutations of X is called the *symmetric group* on X and is denoted $\mathrm{Sym}(X)$:

$$\mathrm{Sym}(X) = \{\phi : X \to X \mid \phi \text{ is bijective } \}.$$

5. Exercises II

Exercise 1. Let $\mathbb{N}$ be the set of natural numbers and let $\mathbb{Z}$ be the integers. Find examples of functions $f : \mathbb{Z} \to \mathbb{N}$ such that:
(a) f is bijective;
(b) f is injective but not surjective;
(c) f is surjective but not injective;
(d) f is neither injective nor surjective.

Exercise 2. Let $\mathbb{N}$ be the set of natural numbers. Let A be a subset of $\mathbb{N}$ given by $[50, 70] \cap \mathbb{N}$, where $[50, 70]$ is the closed unit interval of real numbers between 50 and 70.

Define a function $f : \mathbb{N} \to \mathbb{N}$ by $f(n) = 3n$. Note that A is in both the domain and the codomain of f.
(a) Find the image $f[A]$.
(b) Find the preimage $f^{-1}[A]$.
(c) Show that f is injective.
(d) Show that f is not surjective.

Exercise 3. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3 - 6x^2 + 11x - 3$. Find $f^{-1}[\{3\}]$.

Exercise 4. We would like to define a function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}$ by $(p, q) \mapsto \frac{p}{q}$. Unfortunately, this does not make sense. Fix the problem, and show that the resulting function is surjective but not injective.

Exercise 5. We would like to define a function $f : \mathbb{Q} \to \mathbb{Z}$ by $\frac{p}{q} \mapsto pq$. Unfortunately, this is not "well-defined". Figure out what this means and fix the problem. Is the resulting function injective?

Exercise 6. Let $f : X \to Y$ be a function and let $A, B \subset X$.
(a) Show that $f[A \cup B] = f[A] \cup f[B]$.
(b) Show that $f[A \cap B] \subset f[A] \cap f[B]$.
(c) Give an example where $f[A \cap B] \neq f[A] \cap f[B]$.

Exercise 7. Let $f : X \to Y$ be a function and let $C, D \subset Y$.
(a) Show that $f^{-1}[C \cup D] = f^{-1}[C] \cup f[D]$.
(b) Show that $f^{-1}[C \cap D] = f^{-1}[C] \cap f[D]$.

Exercise 8. Let $f : X \to Y$ and $g : Y \to Z$ be surjective functions. Show that $g \circ f$ is surjective.

Exercise 9. Let $f : X \to Y$ and $g : Y \to Z$ be functions.
(a) Show that if f is surjective and $g \circ f$ is injective, then g is injective.
(b) Give an example where $g \circ f$ is injective but g is not.
(c) Show that if g is injective and $g \circ f$ is surjective, the f is surjective.
(d) Give an example where $g \circ f$ is surjective, but f is not.

Exercise 10. Let $f : X \to Y$ be a function.
(a) Show that f is surjective if and only if there exists $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$.
(b) Show that f is injective if and only if there exists $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$.

Exercise 11. Let X be a set and let $\phi, \psi \in \mathrm{Sym}(X)$. Show that $\phi \circ \psi \in \mathrm{Sym}(X)$.

Exercise 12. Let X be a set containing n elements. Try to count the number of functions in $\mathrm{Sym}(X)$.

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